ART. XIII.—*An Attempt to Simplify and Generalise the Theory and Practice of Railway Curves.* By WILLIAM HILL, Esq., C.E.

(With a Plate.)

[Read before the Institute 14th September, 1859.]

MR. PRESIDENT AND GENTLEMEN,—

If we suppose two places to be on the same level, and that they are united by a canal, then the trace of the surface of the water, on a vertical plane, passing through the terminal points of the canal and the centre of the earth, will be a great circle of the globe; but if the two places are not equidistant from the earth's centre, then in its course the canal, proceeding from the higher to the lower station, would have to be traced by concentric arcs, successively of shorter radius, until the minute (in comparison with the radius of the earth) but important difference of level was made in the surface of the canal at the two places. The steps (or locks) by which the descent is effected are each several feet deep; whereas if a railway connected the same two places, it would be made to descend by imperceptible steps, and its longitudinal profile would be a curved line, called the gradient. On paper this curve is, for obvious reasons, drawn as a straight line. A railway can seldom, however, be carried throughout upon one gradient, and a track is selected with regard to the economy of gradients, from point to point of elevation and depression, and impracticable ground is evaded by a judicious se lection of curves, so as to minimize on the plan its departure from a direct trace from point to point. To enable us to do this, so as to avoid obstructions and "lose ground" as little as possible, in short, to economise curves, is the object of the present attempt to generalise the subject, and to simplify the

consideration of the capabilities of the ground—what curves are possible, first for the evasion of obstructions, and next for the recovery of the direct course, from point to point, predetermined to be passed through.

I have treated the subject analytically, and have aimed at a tangible set of simple deductions from the data, to be fixed in the mind, and as applying to all cases, so as in a great degree to supersede theorising in the field, and to have the mind as free as possible for viewing leisurely and coolly the practical and physical difficulties of execution.

The track of a railway may be considered as a system of curves and their tangents, each curve to the right or left necessitating an early reverse curve; for the further a divergence is continued, the more there is to rectify by the return curve. It has least to do when the two reciprocating arcs are contiguous.

It is proposed to prove that the locus of the point of contact, or of osculation of these two arcs, and the locus also for a compound curve, is the arc of a circle, which may be called the "arc of contact," and that where both are possible the locus of both is the same arc.

The formulæ for finding the radius and position of the tangents of the arc of contact are simple, so that the arc may be set out very readily; and it will be seen that it at once affords a key to the capabilities of the ground. I propose to consider other formulæ connected with or independent of the arc of contact.

The Arc of Contact for Simple and Compound Curves.

Investigation No. I. (of the "arc of contact," or locus of the osculating point (A) of every possible **S** or compound curve, from a given railway tangent at B to a railway tangent at D.

It is first to be shown that the locus of the osculating point is the arc of a circle, in either case.

Figure M shows the usual position of the parts in an **S** curve, the angles α and β opening each to the right out of BD, to an observer stationed on the angular points B and D respectively. Figures N and O show a possible arrangement of the parts, the angles α and β opening right and left respectively out of DB, so that, considered as opening to the right, β in each figure (N and O) is, evidently, as determining the sign ($\pm$) of its sine, cosine, &c., to be placed or taken be, in the fourth quadrant. Figure P will be sufficient for the

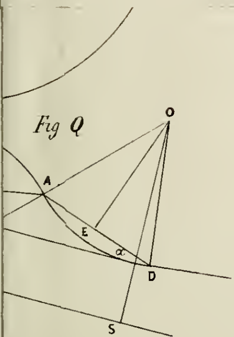
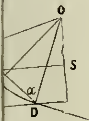


Fig Q



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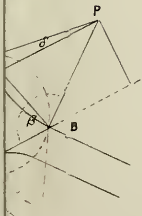

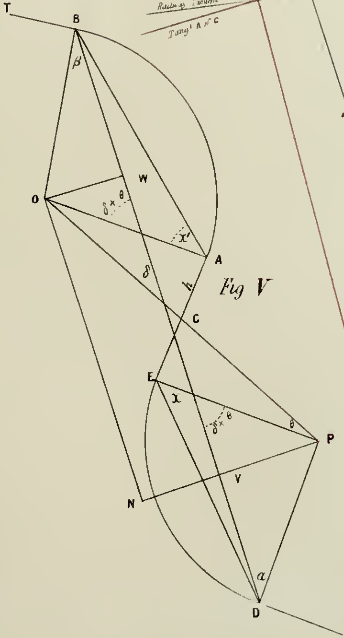

Fig V

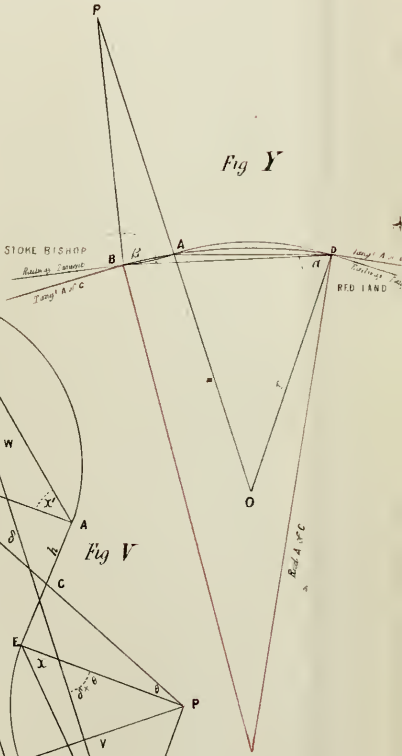

Fig Y

...ls Radii are **BLUE**

Radii & Tangents **RED**.

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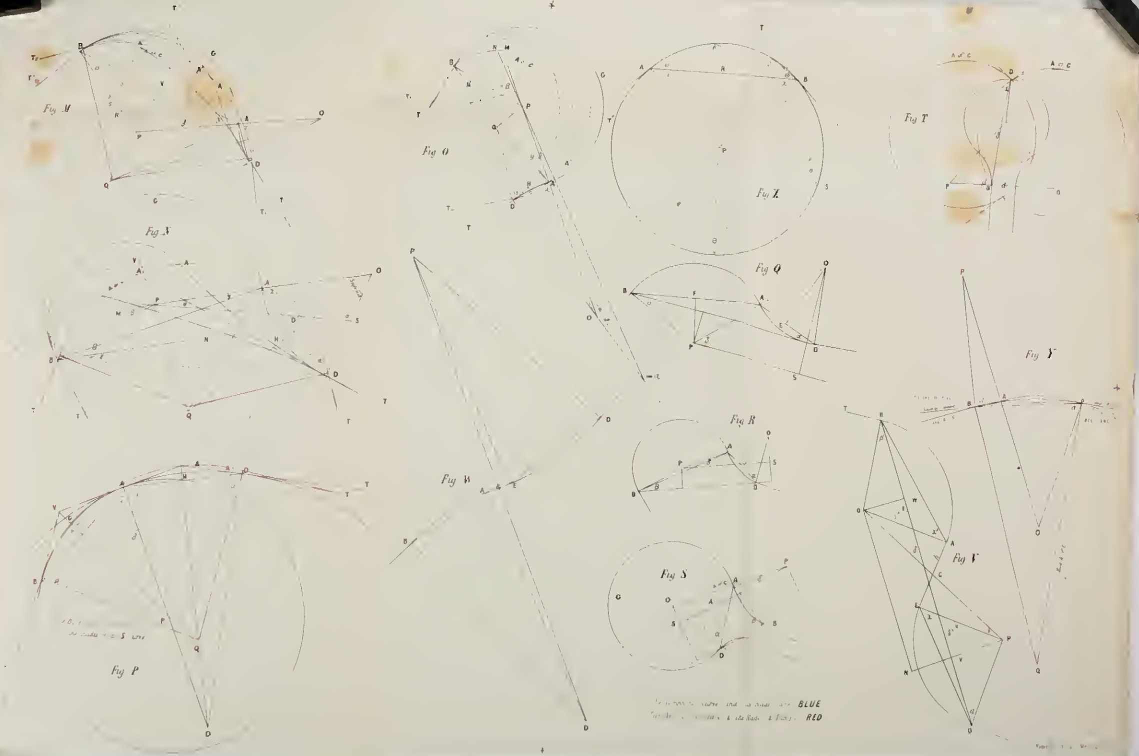

Fig M
Fig O
Fig Z
Fig T
Fig Q
Fig Y
Fig R
Fig W
Fig V
Fig S
Fig P
BLUE
RED

investigation as regards a compound curve, there being not the same variety of appearance in the figures assumed by this as by the less "prehensible" and eel-like S curve, to all of which, however, the same trigonometrical principles apply. We have only to be attentive to the sign positive or negative of the gonometrical ratios, and for that purpose, and that only, to estimate all the angles as opening on the same hand right or left; and as to their angular quantity (as of α and β), to take them simply as they appear by their necessary designation in the figures. The repetition of this precept may be necessary.

To prove that the locus of osculating point is a circular arc.

Note—That α and β are the angles of intersection of the *radii* of the two arcs, at D and B, with BD.

FIG. M.

The $\angle$ OAD = half supt. of O

or, $OAD = \frac{\alpha + \delta}{2} = ADB + \delta$

(by Ext. $\angle$ = two inter. & opposite)

$\because ADB = \frac{\alpha - \delta}{2}$

and the $\angle$ PBA = half supt. of P

or, $PBA = \frac{B + \delta}{2} = ABD + \beta$

$\because ABD = \frac{\delta - \beta}{2}$

and $\because$ The angle BAD = supt. $\frac{\alpha - \beta}{2}$

FIG. N.

$\left.\begin{array}{l} BN \\ PS \end{array}\right\}$ are parallels to $\left\{\begin{array}{l} PO \\ BD \end{array}\right.$

(Either an S or C.)

The $\angle$ OAD = half supt. of O

or, $OAD = \frac{\alpha + \delta}{2} = ADB + \delta$

(for ADB = AD′P; and OAD is the exterr. $\angle$ of the Δ, AD′P; and AD′P and δ are the interr. and opposite $\angle$cs.)

$\therefore ADB = \frac{\alpha - \delta}{2}$

$ABP = PAB = \frac{MPB}{2} = \frac{\beta - \delta}{2}$

(as shown by construction of the figure)

MPB or $\beta - \delta$ being exterr. $\angle$ of the Δ, ABP and A and B being equals.

But $ABP = \beta - DBA = \frac{\beta - \delta}{2}$

$\therefore DBA = \frac{\delta + \beta}{2}$

and $\therefore$ The angle BAD = supt. of $\frac{\alpha + \beta}{2}$

FIG. O.

$$POS = a - \delta$$

$$ADO = \frac{POS}{2} = \frac{a - \delta}{2}$$

$$\because ADB = a - ADO = \frac{a + \delta}{2}$$

$$ABP = \frac{MPB}{2} = \frac{\beta + \delta}{2}$$

$$\because ABD = \beta - ABP = \frac{\beta - \delta}{2}$$

$\because$ The angle $BAD =$ supt. of $\frac{a + \beta}{2}$

FIG. P.

$$ABP = \frac{\beta + \delta}{2} = ABD + B$$

$$\therefore ABD = \frac{\delta - \beta}{2}$$

$$ODA = \frac{a + 180° - \delta}{2} = ADB + a$$

$$\therefore ADB = \frac{180° - a - \delta}{2}$$

$\therefore$ remaining $\angle$, $BAD = 180° - \frac{180° - (a + B)}{2}$.

or, the angle $BAD = 90° \div \frac{a + \beta}{2}$

The same in effect as Fig. M, if the same $\angle$ be taken as a, for $90° + \frac{a + \beta}{2} = 180° - \frac{(180 - a) + \beta}{2}$ *

In like manner it might be proved that the same values respectively attach to the $\angle$ BAD, at whatever other point A′, two arcs, tangt. at B and D respectively, may osculate. If therefore a segment of a circle be described through the points BAD, that is, if there be drawn a segment capable of containing an angle of the proved value of BAD, its arc will be the locus of osculation.

Q. E. D.

Cor.—From these two things, viz., the $\angle$ of segment and the length DB of its chd., the position of the tangents of the arc of contact and its radius are determined. Thus—

For *setting out* this arc on the ground.

BD is the chd., and the *position of the tangents* of the arc of contact is at once determined, for they intersect the line DB at angle (T DB or T′BD) = BAD,—(the latter being the $\angle$ in the alternt. Segmt.)

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* Or it might have been proved by means of Fig. M, that BA‴D has the same value as BAD – put x', y' for the contiguous angles, in Fig. M, of the isosceles triangles whose bases are the chords BA‴. A‴D (as x and y are in Fig. P), and let δ' in Fig. M serve as auxiliary to the proof.

$$\text{Then } x' = \frac{(180° - a) + \delta}{2}$$

$$y' = \frac{(180° - \delta) + \beta}{2}$$

and BA‴D is their sum $= \frac{360° - a + \beta}{2} =$ suppt. of $\frac{a - \beta}{2}$

Value of $\angle$ (A, A′ or A″) in the Segmt.	In the ordinary position of the parts, as in fig. M, for an S curve	BAD = Supt. of $\frac{\alpha-\beta}{2}$
	In the possible arrangement of parts, as in fig. N,O, for an S curve	BAD = Supt. of $\frac{\alpha+\beta}{2}$
	In the case of a compound curve	BAD = $90^\circ + \frac{\alpha+\beta}{2}$

(The same in effect as in fig. M, but that α in this figure is the conpt. of α in fig. M.)

(As proved on preceding page.)

$$\text{The radius A of C} = \tfrac{1}{2}\text{ chd. cosec. }\frac{Q}{2} = \frac{BD}{2}\text{ Cosec. BAD}$$

$$\text{Or, if a table of secants is not at hand, say r. a. of c.} = \frac{\frac{1}{2}\,BD}{\text{Sin. BAD}}$$

A simplification of common method of setting out curves will be explained hereafter.

Use of the arc of contact.—It is observable that the prolongation of one of the railway tangents TD intersects the arc of contact at A″, (this prolongation may be considered, for uniformity's sake, an arc of infinite radius), but that beyond that point the arc of contact is not available for an S curve, nor short of that point for a compound curve. It is evident, however, that if the absence of obstructions allows the choice of any available point on it, as A or A′, the S or compound curve will have greater freedom than the sharp curve A″B.

The length DA″ of the prolongation of the tangent that of the chord A″B and r, the radius of such simple arc, may be found by calculation; thus, (values of A″, B, D, being given in the investigation below.)

The DATA are DB or d and α and β.

$$\text{(I.) } \underline{\text{D}}\text{A}'' = d \text{ sin. } \overline{\text{A}+\text{D}} \text{ cosec. A}''; \quad \text{(II.) } \text{B}\underline{\text{A}''} = d \cos \alpha \text{ cosec. A}''$$

$$\text{and } r = \frac{AB}{2}\text{ cosec. A}''. \quad \text{(III.)}$$

Note—That this observation and these three formulæ apply to the case where one of the railway tangents produced cuts the arc of contact. At this end of the curve is the $\angle$ designated the α in the second of these formulæ.

Investigation of the above three formulæ.

In the Δ A″DB, the $\angle$ A″=BAD, the $\angle$ of the segment. (Its value already stated.)

$\angle$ D $=\alpha - 90^\circ$ in all cases of one arc curve.

$\angle$ B $=$ supt. (A″ + D)

Sin. A'' : sin. B : : d : $A''D = d$ sin. B cosec. A''
Sin. A'' : sin. D : : d : $A''B = d$ sin. D* cosec. $A'' = d$ cos. a cosec. A''.

But $A''=A$ or DAB and $BA''T''=$ supt. $A=\frac{R}{2}$ (fig. M)

and the cosect. of an ∟ = cosect. of its Supt.

$\therefore$ $A''D = d$ sin. (A+D) cosec. A ; $A''B = d$ cosec. a cosec. A

and by rt. ∟d. trigy. $r=\frac{A''B}{2}$ cosec. A.

The choice of Curves.

If the arc of contact be set out merely to the available extent, then an assistant may be directed to move along it, and be signalled from the D or B, to fix an eligible point of osculation. It will then be absolutely necessary to measure only the angle D or B where you stand of the Δ, DAB, of which the base DB and its opposite ∟ A, are already known; we then have all the angles and the base of Δ, DAB, and the difference between a and D = either ∟ at the *base of the isosceles*, whose base is the chd. AD, and equal sides OA, AD and the difference between β and B = either of the *corresponding angles* of the Δ, whose base is AB, and sides the radii AP, PB of the other arc.

The chords are the two remaining sides of ADB—

Sin. A : Sin. D : : BD or d : Chd. AD = d. Sin. D. Cosec. A
Sin. A : Sin. B : : d : Chd. AB = d. Sin. B. Cosec. A

Then $r' = \frac{AD}{2}$ Sec. ADO* and $r'' = \frac{AB}{2}$ Sec. ABP†

It is thus seen that after the arc of contact is set out the rest is exceedingly simple, and in obtaining the elements for that arc no difficulty presents itself, since the angle A of the segment, which is the key to the whole, is plainly enough stated at pages 126-7, and in terms of the known angles a and β; and the ∟ at base of each isosceles on this page.

N.B.—For distinction the arc of contact, and the position of its tangent, may be marked by arrows tipped with red cloth.

One radius given, to find the other radius, in Ogee and Compound Curves.

It will be useful to run through the process by which, with the aid of the arc of contact, we may find a radius for the second arc, the radius of the first being given.

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* Cos. a = sin. D = sin. $a-90°$

† ADO and ABP are the ∟es at bases respectively of the isosceles Δes.

(In this investigation, let the centre of the arc that is "without" the arc of contact be designated by O, and its chd. by AD, and let a be at that end of BD, as in all the figures.)

In all cases of **S** curves,

$\angle$ OAQ $= (\bar{A} + a) - 90°$

PAQ $= 270° - (\bar{A} + a)$

And in compound curves,

OAQ and PAQ are the same $\angle = 90° + a - \bar{A}$*

and r. arc of contact is common to the two Δs, OAQ, PAQ

Then,

If r', the radius OA, be given, or If r'', the radius AP, be given,

Then in Δ OAQ — Given two sides, viz., r' and r. arc of contact, and included $\angle$ OAQ— To find $\angle$ AOQ, the $\angle$ of the segment opposite to AHD (being half the $\angle$ at the centre O).

Then in Δ PAQ — Given two sides, viz., r'' and r. arc of contact, and included $\angle$ PAQ To find $\angle$ APQ, whose supt. is APV, the $\angle$ of the segment opposite to AGB

(by the well-known rule $a + b : a - b :: \tan. \frac{A+B}{2} : \tan. \frac{A-B}{2}$

and thus finding half-difference of the unknown $\angle$es, and knowing half their sum, we get AOQ) which is the half sum *plus* half difference if its opposite side (r. arc of contact) is *greater* than r' or r'' respectively—(and *minus*, if *less.*)

At this $\angle$ AOQ, or At $\angle$ APY,

The chd. AD must be deflected from the Ry. tangent. The *length* of chd. AD $= 2\,r'$ sin. AOQ.

The chd. AB must be deflected from the Ry. tangt. The *length* of chd. AB $= 2\,r''$ sin. APV.

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* In Figs. M, N, ODQ = OAQ $= a +$ compt. of supt. of A
In Fig. O $= a +$ compt. A $\Big\} = A + a - 90°$

and PAO = supt. of OAQ $= 270° - (A + a)$

In a compound curve (Fig. P) ODQ = OAQ $= a -$ comp. supt. $A = a + 90° - A$.

Next having found

The chd. AD and its position
Let x be the $\angle$ at base AD of isosceles ADO (compt. of AOQ)
Then in Δ BAD, D,($=a-x$) and A being known, we have given two sides AD, DB and all the angles to find AB.
Lastly in the isosceles APB we have AB and $\angle$ ABP, or $y=\beta \pm \text{ABD}$*, and $\frac{\text{AB}}{2}$ Sect. $y=r''$
In a compd. curve $y = \beta + \text{ABD}$

Or, Chd. AB and its position.
Let y be the $\angle$ at base AB of isosceles ABP (compt. of APQ)
Then in Δ BAD, $\text{B}=\beta \backsim y$; (i.e.) the difference between β and y and A being known, we have given two sides AB, BD and all the angles to find AD.
Lastly in the isosceles AOD we have AD and $\angle$ ADO or $x = a - \text{ADB}$ and $\frac{\text{AD}}{2}$ Sect. $x=r'$
In a compd. curve $x = a + \text{ADB}$

Moreover when x or y is known, then $\left.\begin{matrix} y=\text{BAD}-x \\ \text{or } x=\text{BAD}-y \end{matrix}\right\}$ in a compound curve.

It has thus been attempted to simplify and generalise the various possible cases of **S** and compound curves, and to present the whole at one view, in order, as has been premised, to anticipate practical difficulties in the field by such general theory, and provide a starting point for their solution.

It may be excusable here to point out the importance of keeping these few essential points steadily in view, viz., the distance BD and the angles a and β (alpha and beta) are of course *given*. x (and y for distinction when two arcs are concerned) is the angle at the base of each isosceles Δ (see fig. X), and this angle is the compliment of the several $\angle$es marked, in the figure, Θ (the capital *theta*) viz.: 1st, of the $\angle$ of deflection of the chord from the tangent of the arc. 2nd, of any angle in the alternate segment. And 3rd, of half the $\angle$ at the centre (i.e.) it is the complement of the angle of a right-angled Δ, whose hypothenuse is the radius of the arc, and whose opposite side is half the chord. In short, $x=90-\text{TAB}=90°-\text{ASB}=90°$—APR.

The $\angle$ BAD is easily made out (see page 127). Also, AOQ and PAQ, when one radius is given (see page 129); and thence to find AOQ, when the radius OA is given, and APV when the radius PA is given, and their compts. severally x and y at bases of isosceles.

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* In a compound curve, and in the ordinary figure M of an **S** curve, $y = \beta + \text{ABD}$, the minus sign occurs when (as in fig. N, O) the two centres of an **S** curve are on the same side of DB.

In fig. P, an **S** curve might be adopted up to A″ from D. Let the blue lines be the radii of such; then the $\angle$es OA″Q, PA″Q, the required $\angle$ AOQ or APQ, and chord AD or AC, will be as in fig. MNO. The (angle) DAB being the same in either case.

The explicit directions for setting out (page 137) with the simple example given on page 139, will make every thing else sufficiently comprehensible, and it is believed that it will all be found easy after a little practice in the field, in the ordinary position of the parts certainly, but it has been deemed most desirable to attempt to lay down one general plan to provide for all possible cases.

Ogee Curves of Common Radius.

Investigation of formulæ for finding the radius that shall be common to the two arcs of an S curve, together with the length and position of the chords.

It will be proved that

I. $\text{Sin. } \delta = \dfrac{\text{sin. } \alpha + \text{sin. } \beta}{2}$
(and *only* so when the radius is common)

II. $r = \dfrac{\text{DB}}{\text{cos. } \alpha + \text{cos. } \beta + 2 \text{ cos. } \delta}$

III. Chord DA $= 2\, r \text{ cos. } \dfrac{\alpha + \delta}{2}$
deflected at an $\angle = 90° - \dfrac{\alpha + \delta}{2}$
from the tangent.

IV. Chord AB $= 2\, r \text{ cos. } \dfrac{\beta + \delta}{2}$
at $90° - \dfrac{\beta + \delta}{2}$

It has only to be remembered, as far as these formulæ are concerned, that, as before observed, the proper station to consider an angle is the angular point (as B or D) and that the angle α or β is then to be estimated in determining the sign (plus or minus) by the "*quadrant it is in*;" supposing the angle to be the degrees passed over by the radii OD, PB, severally and turning out of the same line DB on the same hand, right or left. The usual conventional direction is to the right. Thus, then, an angle in the 3rd or 4th quadrant has its *sine* minus, and one in the 2nd or 3rd quadrant has its *cosine* minus.

The numerical values of these ratios and their degrees are to be taken simply as those of the angles designated α and β *in the figures.* Thus, in the first place, in fig. R, α is in the 2nd quadrant, and β in the 4th quadrant.

$\therefore$ The sine of α is $(+)$; its cosine $(-)$
Sine of β is $(-)$; cosine $(+)$

so that in formulæ I. the sign $(+)$ of sin. β is reversed, and becomes $(-)$; and in formulæ II. cos. α becomes $(-)$; whilst, secondly, as regards their value in degrees, in formulæ III. and IV. α, β and δ merely represent the angular *quantities so designated in the figures,* so that the angular quantity $\dfrac{\alpha + \delta}{2}$ and $\dfrac{\beta + \delta}{2}$ are necessarily less than a right angle, because they are each half of two angles of a triangle.

With this caution, these formulæ are universally applicable, or, as it is technically called, "general."

Investigation of Formula No. I.

Let PS be drawn parallel to BD. Then, in Fig. Q,
$OS = OF + EP$, but $OS = OP \sin. \delta + 2\, r \sin. \delta$
and $OF + EP = OD \sin. a + PB \sin. \beta = r\,(\sin. a + \sin. \beta)$

$$\therefore\ 2\, r \sin. \delta = r\,(\sin. a \times \sin. \beta) \qquad \therefore\ \text{Sin. } \delta = \frac{\sin. a + \sin. \beta}{2}$$

N.B.—By the trigonometrical law of signs, this formula is true for all values of a and β, which angles are in this figure acute angles, and have all functions (Sin., Cos., Tangt., &c.) positive.

And it is easily shown that what is said above of Fig. R as to the alteration of sign $(\pm)$ in the fourth quadrant, makes sine β *negative* in Fig. R.

Where $OS = OP \sin. \delta = 2\, r \sin. \delta = OF - EP = r\,(\sin. a - \sin. \beta)$

so that the equation *becomes* $\sin. \delta = \dfrac{\sin. a - \sin. \beta}{2}$ in Fig R.

Investigation of Formula No. II.

Fig. Q.

$$BD = r \cos. a + 2\, r \cos. \delta + r \cos. \beta$$

$$\therefore\ r = \frac{BD}{\cos. a + 2 \cos. \delta + \cos. \beta}$$

N.B.—This is true also for all values of a and β, which are in this figure acute angles, and have all functions positive, and in Fig. R it is easily shown that the sign $(\pm)$ of $\cos. a$ is reversed from its being in the second quadrant, for $BD = r \cos. \beta + 2\, r \cos. \delta - r \cos. a$

$\therefore$ The equation *becomes,* $r = \dfrac{BD}{\cos. \beta + 2 \cos. \delta - \cos. a}$ (in Fig. R.)

As to Formulæ III. and IV., it is only required to show that—the triangles DOA and BPM being isosceles, their equal sides (τοι ισοσκελοι) being the equal radii, and their equal angles, at the bases DA, BA, being severally a mean between a and δ and β and δ—if then each be bi-sected by the perpr. from apex upon base, into two right-angled triangles, of which half the chord is the adjacent side to the $\angle$ at the base, and of which, radius is the hypothenuse; then

$$\therefore\ \frac{\text{adj. si.}}{\text{hyp.}} = \cos. \quad \therefore\ \cos.\ (\angle \text{ at base}) = \frac{\tfrac{1}{2}\text{ chd.}}{r} = \frac{\text{chd.}}{2\, r}$$

$$\therefore\ \text{chd.} = 2\, r \cos.\ (\angle \text{ at base.})$$

(*i.e.*) chd. AD and AB $= 2\, r \cos. \dfrac{a + \delta}{2}$ and $2\, r \cos. \dfrac{\beta + \delta}{2}$, severally.

and the angles deflected out of the direction of tangents are evidently the compliments severally of these angles at the bases aforesaid. In a compound curve, the radii approximate equality as the angles α and β do so; when $\alpha = \beta$ then the two arcs merge into one simple arc.

Note.—The extraordinary figure S is drawn to show an S curve DGB of equal radii, to unite an *arched* unto a *straight* tangent, the arc of contact being only a small portion of a very small circle, on which osculating points for two arcs of *differing* radii might, however, be chosen between D and A″ on the arc of contact; and figure T, by way of showing the general applicability of the formulæ, is drawn with two symmetrical S curves of equal radii, to which the above formulæ are applicable; and the arc of contact of each is also drawn, to show that the principle of *that* also applies to all sorts of symmetrical figures formed by the ogee curve.

Intermediate Straight.

Investigation of a rule for the direct calculation of a common radius, to admit a given length of intermediate tangent.

Fig. V.

$$CP = \sqrt{r^2+h^2} \qquad \therefore\ OP = 2\sqrt{r^2+h^2}$$

$$PN = r\,(\sin.\ \alpha + \sin.\ \beta)\,;\quad VW = NO = \sqrt{OP^2 - NP^2}$$

Let m be put for $\sin.\ \alpha + \sin.\ \beta = 2\sin.\ \frac{\alpha+\beta}{2}\cos.\ \frac{\alpha-\beta}{2}$

Then $PN = rm$

(Note.—If tangents are parallel $m = 2\sin.\ \alpha$ because $\alpha = \beta$)

$$\overline{VW}^2 = \overline{OP}^2 - \overline{PN}^2$$

and $\overline{VW}^2 - 4h^2 = 4r^2 - m^2 r^2 \qquad \therefore\ r^2 = \dfrac{\overline{VW}^2 - 4h^2}{4 - m^2}$

But (see fig.) $\overline{VW}^2 = d - r\,(\cos.\ \alpha + \cos.\ \beta) =$

Let n be put for $\cos.\ \alpha + \cos.\ \beta = 2\cos.\ \frac{\alpha+\beta}{2}\cos.\ \frac{\alpha-\beta}{2}$

Then $VW = d - rn$

(Note.—If tangents are parallel, then $n = 2\cos.\ \alpha$.)

$$\therefore \ \mathrm{VW}^2 = d^2 - 2\ drn + r^2\ n^2\ ;\ \text{but as } r^2 = \frac{\overline{\mathrm{VW}}^2 - 4\ h^2}{4 - m^2}$$

$$\therefore \ r^2 = \frac{d^2 - 2\ drn + n^2\ r^2 4\ h^2}{4 - m^2}$$

$$\left\{1 - \frac{n^2}{4 - m^2}\right\} r^2 = -\frac{2\ nd}{4 - m^2} r + \frac{d^2 \cdot h^2}{4 \cdot m^2}$$

Let p be put for $d^2 - h^2 = (d + 2\ h)\ (d - 2\ h)$ and cancel com. den. r.

$$4 - (m^2 + n^2)\ r^2 = -2\ nd\ r + p$$

$$\therefore \ r^2 = -\frac{2\ nd}{q} r + \frac{p}{q},$$

if q be put for $4 - (m^2 + n^2)$

$$\therefore \ r = \frac{\sqrt{\overline{nd}|^2 + pq} - nd}{q} \quad \text{(the rule required.)}$$

For the length and position of the chords we have—

$$\text{Sin. } \delta = \frac{\mathrm{PN}}{\mathrm{OP}} = \frac{rm}{2\ r \text{ sec. } \theta} = \frac{m}{2} \text{ cos. } \theta$$

$$\text{Cos. } \theta = \frac{r}{h} \text{ or tan. } \theta = \frac{h}{r}$$

The angles at the bases of the isosceles triangles DPE AOD, are severally the mean between a and $\delta + \theta$ and β and $\delta + \theta$. (See Fig. V.)

$$(i.e.)\ \mathrm{EDP} = \frac{a + \delta + \theta}{2} \quad \text{and} \quad \mathrm{OBA} = \frac{\beta + \delta + \theta}{2}$$

$$\text{Chd. AD} = 2\ r \text{ cos. EDP and AB} = 2\ r \text{ cos. OBA}$$

The chords may be deflected from the tangents at the counpts. respectively of these angles (EDP and OBA).

Note.—If the tangents are parallel, then

$$\text{Sin. } \delta = \frac{\mathrm{PN}}{\mathrm{OP}} = \frac{2\ r \text{ sin. } a}{2\ r \text{ sec.}} = \text{sin. } a \text{ cos. } \Theta$$

Tangent θ being as before $\dfrac{h}{r}$

Example, Fig. V.

Supposing the distance DB or $d = 285$, and the demi-tangent EC or $h = 31$, and that the angle $\alpha = 38° \ 20'$, and the angle $\beta = 29° \ 30'$—Required, the common radius of the two curves, and the length and position of the chords.

α	38° 20′	by 2	0·3010300		0·3010300
β	29° 30′		9·7466237	cos.	9·9189996
			9·9987084		9·9987084
sum	67° 50′				
diff.	8° 50′	by $m =$	0·0463621		0·2187380 by n
half sum	33° 55′ by sin.			by d	2·4548449
half diff.	4° 25′ by cos.			by dn $=$	2·6735829
					2
		$\therefore$ $dn^2 = 222415{\cdot}87$		by $dn^2 =$	5·3471658

by $m^2 = 0{\cdot}0927242$ and $m^2 = 2{\cdot}738268$

by $n^2 = 0{\cdot}4374760$ and $n^2 = 1{\cdot}23801$

$4 - 3{\cdot}976278 = {\cdot}023722$ the denr.

$d = 285$

$2h = 62$

$347 \times 223 = p$

by $q =$	2·3751513		
by 347 $=$	2·5403295		
by 223 $=$	2·3483049		222415·87 $=$ Ansr.
by $pq =$	3·2637857	$pq =$	1835·6326
			224251·5026

2)

by $=$ 5·3507354

$\because$ root $= 473{\cdot}552$ by root $= 2{\cdot}6753677$

$nd = 471{\cdot}61$

The numr. 1·942 by numr. $= 0{\cdot}2882492$

— denr. $= 2{\cdot}3751513$

by $r = \overline{1}{\cdot}9130979$

$\because$ $r = 81{\cdot}865$ Answer.

To calculate the Chords.

$$\text{Tan. } \theta = \frac{h}{r} = \frac{31}{81{\cdot}865} \qquad \begin{array}{r} 1{\cdot}4913617 \\ 1{\cdot}9130979 \\ \hline \end{array}$$

$\because$ $\theta = 20°\ 44'\ 25''$, Tan. $\theta = 9{\cdot}5782638$, cos. $\theta = 9{\cdot}9709123$

$$\begin{array}{rr} \text{by } m - & 0{\cdot}0463621 \\ \hline & 0{\cdot}0172744 \\ \text{by } 2 = & 0{\cdot}3010300 \\ \hline \text{by sin. } \delta = & 9{\cdot}7162444 \end{array}$$

$$\begin{array}{rl} \because\ \delta = & 31°\ 21'\ 6'' \\ \theta = & 20°\ 44'\ 25'' \\ a = & 38°\ 20'\ \text{—} \\ \hline 2) & 90°\ 25'\ 31'' \\ \hline \angle \text{EDP} = & 45°\ 12'\ 45\tfrac{1}{2}'' \\ \hline \end{array} \qquad \begin{array}{rl} & 29°\ 30' = \beta \\ \hline 2) & 81°\ 35'\ 31'' \\ \hline \angle\ \text{ABO} = & 40°\ 47'\ 45\tfrac{1}{2}'' \\ \hline \end{array}$$

Chord ED $= 2\,r$ cos. $45°\ 12'\ 46''$ AB $= 2\,r$ cos. $40°\ 47'\ 46''$

$$\begin{array}{rr} & 0{\cdot}3010300 \\ & 1{\cdot}9130979 \\ & 9{\cdot}8478460 \\ \hline \text{by ED} = & 2{\cdot}0619739 \\ \hline \because\ \text{Chord ED} = & 115{\cdot}34 \end{array} \qquad \begin{array}{rl} 9{\cdot}8791012 & \\ \hline 2{\cdot}0932291 & = \text{by AB} \\ \hline \text{Chord AB} = & 123{\cdot}94 \end{array}$$

Example for Practice.

FIG. W.

Let $d = 150$, the demi-tangent $= 10$
$a = 80°$ $\beta = 74°\ 20'$

Required—The radius, and length, and position of the chords.

Answer—$r = 16405$
EDP $= 79°\ 56'\ 55\tfrac{1}{2}''$
ABO $= 77°\ 36'\ 55\tfrac{1}{2}''$
AB $= 70{\cdot}373$
ED $= 57{\cdot}267$

The "Setting Out" of Arcs on the Ground.

The following remarks are themselves rather "lengthy," but if they tend to shorten and simplify the process they refer to, they will perhaps be read with patience.

If the tangent points and the length and position of the chords have been determined by any of the preceding or other methods, then the setting out of the simple arcs of circles can be proceeded with by either of the well-known methods most adapted to the case.

It will here only be attempted to explain a simplification of the method by the theodolite, first observing that if there be two tangents given on the ground to be united by an arc of a circle, and if no previous fixing exactly of the points of contact has been attempted, the best plan is to take one point—the most convenient—as fixed, on one of the tangents, the point on the other to be afterwards fixed, in subordination to this, and in the following order (supposing that the point of intersection of the tangents is distant or inaccessible). Set up the theodolite at an assumed approximate point on the opposite tangent to that on which the point of contact has been fixed as aforesaid, and direct the instrument along the prolongation of the tangent on which it stands, the index of the vernier being first clamped at 360° of the limb; clamp the lower plate, and bisect the pole or arrow that indicates the direction of the tangent with the vertical web or wire (of the diaphragm, fixed in the interior focus of the telescope, and which is common to the eye-piece and object end). Then unclamp the upper plate, and turn the telescope upon the point fixed upon as aforesaid in the opposite tangent, and read off the angle. Now remove the instrument to aforesaid fixed point, and by the same process read the $\angle$ of divergence of the line of collimation joining this and your previous station from or out of the prolongation of this tangent; take half the sum of these two angles, which at once deflect from your tangent, as the true direction of the chord, whereby you fix the opposite point of contact. Measure the chord.

Note.—It is better, as far as the convenience of reading off, in the further process of setting out, is concerned, if we "fix" the first point on the left hand tangent, so that the instrument is first set up on the right hand tangent, and being removed to the left hand tangent, there remains, to the end, perhaps, of the setting out of the curve.

In whatever way the points are fixed, when that is done, and the length and direction of the chords is ascertained, either by calculation or measurement, then

Let us next consider (Fig. X) that the $\angle$ of deflection of the chord A,B from the produced tangent is the $\angle\theta$ of the alternate segment,

and half the angle at the centre; that it is also the $\llcorner$ of a right-angled triangle, and that half the chord or $\frac{C}{2}$ is its opposite side and that r, or the radius of the curve, is the hypothenuse, and recollecting that in right-angled trigonometry, if we want a side, it is convenient to make the given side radius; that is, it is convenient to put given and required sides into the form of a vulgar fraction, with the given side for the denominator; we should then have this fraction,

$$\frac{\text{hypothenuse}}{\text{opposite side}} = \frac{r}{\frac{c}{2}}$$

Now, according to trigonometrical definitions of sine, tangent, secant, &c.,

$$\frac{\text{hypothenuse}}{\text{opposite side}} = \text{cosec. } \theta = r/\frac{c}{2}$$

$$\because\ r = \frac{c}{2} \text{ cosec. } \theta. \quad \text{By this we find radius.}$$

And the same form of equation applies to the *small* half-chord $\frac{c}{2}$ the radius r of the circle, and the *little* $\llcorner$ θ.

$$\text{viz.}: \quad r = \frac{c}{2} \text{ cosec. } \theta \ldots\ldots (\text{I})$$

$$\text{and } \because \text{ cosec. } \theta = \frac{2\,r}{c} \ldots\ldots\ldots\ldots (\text{II}) \quad \because \text{ Sin. } \theta = \frac{c}{2\,r} \quad \begin{matrix}(\text{II.})\\ a\end{matrix}$$

$$c = 2\,r \text{ sin. } \theta \ldots\ldots\ldots (\text{III})$$

So that θ may be found from its cosec. or sin.; by the former generally, most easily by natural numbers and natural cosecant, because c being arbitrary, may be assumed an easy divisor.

Now if we have a very small table of natural cosecants to only a few degrees extent, and for angles only that may be read on the limb of the theodolite, we may then use the vernier to verify our readings merely, which it will do by showing that the index and the last division on the vernier are in complete coincidence with densions in the limb, by ocular comparison with the intermediate divisions. We have only to consult such a table to see instantly the nearest cosecant to the value in decimals of $\frac{2\,r}{c}$ and then assuming this, which is an angle which can be set off with great certainty and precision, as θ—use the reverse formula, No. III. $c = 2\ r$ sin. θ. By this we find c, which can be done in the field easily by means of table of natural sines, of like extent, &c., as the above-mentioned table of cosecants.

The angle θ is thus made readable to very great accuracy, whilst measuring of the sub-chord c to a nicety is a matter of no difficulty.

The little tables of sines and cosecants might be written on a card for the waistcoat pocket.

Curving of Practice on Durdham Down.

Example for Setting Out.

FIG. Y.

The tangents of an imaginary railway being marked out, the distance BD measured 24·32 chains, a was found to be 68° 50′, and β 92° 57′, measured by the theodolite. An **S** curve being required, it was proposed to find, first, the arc of contact, or "locus" of the various points of possible contact for the two arcs.

Then $r^{\backprime}$ arc of contact $= \frac{d}{2}$ cosec. $\frac{\beta - a}{2}$; or $\dfrac{\frac{1}{2} d^*}{\sin. \frac{\beta - a}{2}} = \dfrac{12{\cdot}16}{\sin. 12° \, 3\frac{1}{2}'}$

$$\begin{array}{r} 92° \; 57' \\ 68° \; 50' \\ \hline 2)\, 24° \; 7' \\ \hline 12° \; 3\tfrac{1}{2}' \\ \hline \end{array} \qquad \begin{array}{rl} \log. \; 12{\cdot}16 = & 1{\cdot}0849336 \\ \log. \sin. \; 12{\cdot}3\tfrac{1}{2} = & 9{\cdot}3199538 \\ \hline \log. \; r^{\backprime} = & 1{\cdot}7649798 \\ \therefore = & 58{\cdot}218 \; r^{\backprime} \end{array}$$

It was next assumed that $c = 1$ chain

$$\text{cosec. } \theta = \frac{2 \, r^{\backprime}}{c} = \frac{116{\cdot}436}{1{\cdot}}$$

The nearest cosecant to 116·436 on our card was the cosecant of 30′, whose sine (as found on the card) is ·0087265.

We adopt 30′ as θ, and, using the reverse equation, say $c = 2 \, r^{\backprime}$ sin. 30′ $= 116{\cdot}436 \times {\cdot}0087265 = 1{\cdot}016$

The arc of contact was then set out by radiating the angle θ 30′ from the point D, deflected from the prolongation of tangent to the extent of sub-chord 1·016 chain, and the first point was thus fixed; and then the same angular quantity 30′ was subtended by the like distance from this first point, and the ground being clear, the whole arc was set out without removing the theodolite from D.

The arc of contact was set out for practice merely, although it served the purpose of verifying the formulæ and as a check on our

* Having no table of log. cosecants, the value of $r^{\backprime}$ in terms of sine was preferred.